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On the mixed initial Dirichlet–Neumann problem for the wave equation in the circle

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Abstract. The method of integral equation formulation, proposed by Eckhardt and El Sheikh, is extended to solve an even initial mixed problem for the first time. The truncation of the integral equation or, equivalently, the infinite homogeneous algebraic system to which the problem is converted, is justified. In addition, having carried out the progression of the procedures right to the numerical results at several cases, a clarified and comprehensive idea about the usefulness of the method is revealed.

1. Introduction

In 1987, Eckhardt and El Sheikh [1] proposed a generalization of the usual Fourier method to solve initial mixed boundary value problems. This work was highly influenced by the method previously proposed by Chersky [2] for solving mixed stationary problems and which had already been widely applied to solutions of problems in several branches of mathematical physics [3]. Both methods in [1] and [2] are based on defining the extension of the Dirichlet condition compatible with the Neumann data where they are imposed. This is achieved through reduction to a singular integral equation. Since the boundary conditions in the case of an initial problem are usually set equal to zero, the Dirichlet–Neumann initial problem is viewed as a natural generalization of both the uniform Dirichlet and Neumann initial problems as two limiting cases. This generalization manifested itself throughout the procedures of [1], and it will be naturally realized in this work too. Furthermore, as it is a Fourier method which consists of defining the eigenvalues and eigenfunctions of the corresponding Sturm–Liouville problem, the method proposed in [1] has the advantage of leading to a better insight into the structure of the solutions of the underlying problems in contrast to purely numerical methods which ignore theoretical knowledge about the problem. Nevertheless, so far it could only have been applied to odd problems. The reason for this is explained in section 2.

In this work, the even Dirichlet–Neumann initial problem for the wave equation in a circular region is considered. Physically, this problem is the mathematical formulation for the evolution of a vibrating circular membrane under the influence of an initial displacement, with a moving clamped arc of its circumference, while the complementary arc remains fixed. A similar technique to that developed in [1] is followed. The separation of variables leads in turn to a Sturm–Liouville problem of the Dirichlet–Neumann type which is formulated as a singular integral equation with Cauchy's kernel, but now involving an additional term, because for an even function the zero Fourier component does not, in general, vanish. By converting the singular integral equation to an infinite homogeneous system of algebraic

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equations, we show how the above additional term gives rise to cumbersome technical difficulties but which are now finally resolved. The truncation of the latter infinite system eventually determines the small eigenvalues as well as their corresponding eigenfunctions.

In [1], the application of the truncation method to the infinite homogeneous algebraic system was plausibly assumed to be useful in view of the fact that by retaining a greater number of first terms from the Fourier representation of a continuous function as an approximation, the effect of the excluded next term is reduced. In section 5 of this paper, it has been shown that for a given eigenvalue the homogeneous integral equation (algebraic system) to which the problem is formulated is equivalent to an inhomogeneous one (algebraic system) and, therefore, the truncation could eventually be justified. This represents the second principal result achieved in this work.

Finally, the numerical experiments confirm the validity of the method. The eigenvalues designated by means of the truncation method reach their practically infinite precision at suitable orders. As the problem coincides with the uniform Dirichlet case, i.e. when the arc on which the Neumann condition is imposed vanishes, the eigenvalues are those obtained long ago for the latter problem. Thereafter they start to vary monotonically to finally become those of the pure Neumann problem as the above-mentioned arc expands over the whole circumference of the circle. The Fourier components retained on carrying out the truncation(s) can be obtained precisely at sufficiently large orders. As for the excluded components, they can be obtained through a valid approximation. These last results are illustrated graphically.

2. The problem and its reduction to a homogeneous discrete problem

Our point of departure is the dimensionless wave equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = \frac{\partial^2 U}{\partial \tau^2} \qquad (r < 1, |\theta| \leqslant \pi; \tau > 0) \tag{2.1}$$

together with the boundary condition

$$\begin{aligned} &U(1,\theta;\tau) = 0 \qquad c \leqslant |\theta| \leqslant \pi \\ &\frac{\partial U(1,\theta;\tau)}{\partial r} = 0 \qquad |\theta| < c \end{aligned} \tag{2.2}$$

and

$$|U(r,\theta;\tau)| < \infty. \tag{2.3}$$

In addition, the function U should also satisfy the initial conditions

$$\frac{\partial U(r,\theta;0)}{\partial \tau} = 0 \tag{2.4}$$

and

$$U(r,\theta;0) = f(r,\theta) \tag{2.5}$$

where $f(r,\theta)$ is even with respect to θ and a continuous function, and $f(1,\theta) = 0$,

$$c \leqslant |\theta| \leqslant \pi. \tag{2.6}$$

The bounded solution of equation (2.1) satisfying the initial conditions (2.4) and (2.5) can be thought of in the form

$$U(r,\theta;t) = \sum_{i=1}^{\infty} c_i S(r,\theta;\gamma_i)\cos\gamma_i\tau. \tag{2.7}$$

In order for the other conditions (2.2) to be satisfied as well as maintaining the boundedness of the solution, γ_i and $S(r, \theta; \gamma_i)$, $i \in N$, should be the eigenvalues and their corresponding eigenfunctions of the mixed Sturm–Liouville problem consisting of the Helmholtz equation

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \gamma^2 S = 0 \tag{2.8}$$

together with conditions (2.2) and (2.3) in which U is replaced by S. In section 3, it will be shown that the spectrum of this finite problem is discrete as in the case of a uniform problem with linear geometry, namely those values of γ at which the homogeneous algebraic system (3.13) (truncated at a sufficiently large order) has a non-trivial solution, whence the eigenvalues were immediately denumerated in expression (2.7). Moreover, following the same pattern as in the case of a uniform Sturm–Liouville problem, it is a simple matter to show that the eigenfunctions corresponding to different eigenvalues of this mixed problem are orthogonal. Consequently, the coefficients c_i in expression (2.7) are simply defined to be

$$c_i = \frac{\int_{-\pi}^{\pi}\int_0^1 f(r,\theta)S(r,\theta;\gamma_i)r\,\mathrm{d}r\,\mathrm{d}\theta}{\int_{-\pi}^{\pi}\int_0^1 S^2(r,\theta;\gamma_i)r\,\mathrm{d}r\,\mathrm{d}\theta}. \tag{2.9}$$

The mixed condition (2.2) can be replaced by the two uniform conditions

$$S(1,\theta;\gamma) = \varphi_-(\theta;\gamma) = \begin{cases} \text{undetermined} & \text{when } |\theta| < c \\ 0 & \text{when } c \leqslant |\theta| \leqslant \pi \end{cases} \tag{2.10}$$

$$\frac{\partial S(1,\theta;\gamma)}{\partial r} = \varphi_+(\theta;\gamma) = \begin{cases} 0 & \text{when } |\theta| < c \\ \text{undetermined} & \text{when } c \leqslant |\theta| \leqslant \pi \end{cases} \tag{2.11}$$

which should be compatible. It should be noted that for any square integrable function, $\varphi_-(\theta)$, the solution of the Dirichlet problem (2.8) and (2.10) can be written in the form

$$S(r,\theta;\gamma) = \sum_{n=-\infty}^{\infty} \Phi_{n-}\frac{J_{|n|}(\gamma r)}{J_{|n|}(\gamma)}\mathrm{e}^{\mathrm{i}n\theta}. \tag{2.12}$$

Here and henceforth, $\Phi_{n\pm}$ are the complex Fourier components of the functions $\varphi_\pm(\theta)$

$$\Phi_{n\pm} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_\pm(\theta)\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta \qquad \varphi_\pm(\theta) = \sum_{n=-\infty}^{\infty}\Phi_{n\pm}\mathrm{e}^{\mathrm{i}n\theta}. \tag{2.13}$$

Analogously, the solution of the Neumann problem (2.8) and (2.11) for an arbitrary square integrable $\Phi_+(\theta)$ is simply

$$S(r,\theta;\gamma) = \sum_{n=-\infty}^{\infty}\Phi_{n+}\frac{J_{|n|}(\gamma r)}{|n|J_{|n|}(\gamma) - \gamma J_{|n|+1}(\gamma)}\mathrm{e}^{\mathrm{i}n\theta}. \tag{2.14}$$

Thus, to ensure the compatibility of conditions (2.10) and (2.11), it is necessary and sufficient that solutions (2.12) and (2.14) become identical. In other words, the eigenvalues of the above mixed Sturm–Liouville problem are those for which the discrete problem

$$|n|\Phi_{n-}(\gamma) - Q_{|n|}(\gamma)\Phi_{n-}(\gamma) = \Phi_{n+}(\gamma) \tag{2.15}$$

is satisfied, where

$$Q_{|n|}(\gamma) = \frac{\gamma J_{|n|+1}(\gamma)}{J_{|n|}(\gamma)} = \mathrm{O}(|n|+1)^{-1} \tag{2.16}$$

and the functions $\varphi_\pm(\theta;\gamma)$ are restricted as shown on the right-hand side of equations (2.10) and (2.11). Because of its homogeneity, the compatibility equation (2.15) may have solutions only at discrete values for the parameter γ. Denoting by $\{\Phi_{n\pm}(\gamma_i), n \in Z\}$, the solution of equation (2.15) corresponding to the eigenvalue γ_i, the corresponding eigenfunction of the mixed Sturm–Liouville problem can easily be written down

$$S(r,\theta;\gamma_i) = \frac{\Phi_{0-}(\gamma_i)}{J_0(\gamma_i)} J_0(\gamma_i r) + 2\sum_{n=1}^{\infty} \frac{\Phi_{n-}(\gamma_i)}{J_n(\gamma_i)} J_n(\gamma_i r)\cos n\theta. \tag{2.17}$$

Here, use has been made of the relation

$$\Phi_{n-}(\gamma_i) = \Phi_{-n-}(\gamma_i) \tag{2.18}$$

resulting from the even property of the problem. In view of equation (2.15), the solution (2.17) is unique whether it is obtained by using either (2.12) or (2.14).

3. Integral equation formulation of the problem and its reduction to an algebraic system of equations

On performing the inverse Fourier transform to expression (2.15), we get

$$\frac{1}{\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{-\pi}^{\pi} \frac{\varphi_-(t;\gamma)}{1-\mathrm{e}^{\mathrm{i}(\theta-t)}}\,\mathrm{d}t - \sum_{n=-\infty}^{\infty} Q_{|n|}(\gamma)\Phi_{n-}(\gamma)\mathrm{e}^{\mathrm{i}n\theta} = \varphi_+(\theta;\gamma) \tag{3.1}$$

where we have used the relation [1–3]

$$\sum_{n=-\infty}^{\infty} |n|\Phi_{n-}(\gamma)\mathrm{e}^{\mathrm{i}n\theta} = \frac{1}{\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{-\pi}^{\pi} \frac{\varphi_-(t;\gamma)}{1-\mathrm{e}^{\mathrm{i}(\theta-t)}}\,\mathrm{d}t.$$

Restricting equation (3.1) to the interval $(-c, c)$ where $\varphi_+(\theta;\gamma) = 0$ and recalling that $\varphi_-(\theta;\gamma) = 0$ on $c \leqslant |\theta| \leqslant \pi$, we obtain through integration

$$\frac{1}{\pi \mathrm{i}} \int_{-c}^{c} \frac{\varphi_-(t;\gamma)}{1-\mathrm{e}^{\mathrm{i}(\theta-t)}}\,\mathrm{d}t + Q_0(\gamma)\Phi_{0-}(\gamma)\theta - {\sum_{n=-\infty}^{\infty}}' Q_{|n|}(\gamma)\Phi_{n-}(\gamma)\frac{\mathrm{e}^{\mathrm{i}n\theta}}{\mathrm{i}n} = \alpha \tag{3.2}$$

where α is a constant, the prime over the summation symbols indicates that the value $n = 0$ is not included. The constant α is to be designated in such a way that the singular integral equation with the Cauchy kernel (3.2) possesses a bounded solution $\varphi_-(\theta;\gamma)$ [4, p 257]. However, it will turn out that this bounded solution does not depend on α. The solution can be written in the form

$$\varphi_-(\theta;\gamma) = R(\theta)\left[Q_0(\gamma)\Phi_{0-}(\gamma)\bar{I}(\theta) - {\sum_{n=-\infty}^{\infty}}' Q_{|n|}(\gamma)\frac{\Phi_{n-}(\gamma)}{n} I_n(\theta) + \mathrm{i}\alpha I_0(\theta)\right] \tag{3.3}$$

where

$$\bar{I}(\theta) = \frac{\mathrm{i}}{\pi}\int_{-c}^{c} \frac{\xi \mathrm{e}^{\mathrm{i}\xi}\,\mathrm{d}\xi}{R(\xi)(\mathrm{e}^{\mathrm{i}\xi} - \mathrm{e}^{\mathrm{i}\theta})} \tag{3.4}$$

$$I_n(\theta) = \frac{1}{\pi}\int_{-c}^{c} \frac{\mathrm{e}^{-\mathrm{i}(n+1)\xi}\,\mathrm{d}\xi}{R(\xi)(\mathrm{e}^{\mathrm{i}\xi} - \mathrm{e}^{\mathrm{i}\theta})} \tag{3.5}$$

$$R(\theta) = \lim_{\substack{z\to \mathrm{e}^{\mathrm{i}\theta}\\ |z|<1}} \sqrt{(z-\mathrm{e}^{\mathrm{i}c})(z-\mathrm{e}^{-\mathrm{i}c})}. \tag{3.6}$$

The integrals $\bar{I}(\theta)$ and $I_n(\theta)$ are to be understood in the sense of the Cauchy principal value and are not to be confused with modified Bessel functions. For $n > 0$, [3, p 215], we have

$$I_n(\theta) = -\mathrm{e}^{\mathrm{i}(n-1)\theta} \sum_{j=0}^{n-1} \frac{\mathrm{e}^{-\mathrm{i}\theta j}}{2^j} \sum_{m=0}^{j} \frac{(2m-1)!!}{m!(j-m)!} \frac{[2(j-m)-1]!!}{\mathrm{e}^{\mathrm{i}mc}\mathrm{e}^{-\mathrm{i}(j-m)c}} \qquad n > 0 \tag{3.7}$$

and

$$l_0(\theta) = 0. \tag{3.8}$$

Further, it has already been shown [1] that

$$I_{-n}(\theta) = -\mathrm{e}^{-\mathrm{i}\theta} I_n(-\theta). \tag{3.9}$$

In relation (3.7), the notation $(2m-1)!! = (2m-1)(2m-3)\ldots 5.3.1$ is used, and the convention $0!! = (-1)!! = 1$ is adopted. The integral $\bar{I}(\theta)$ is rather difficult to deal with since in this case, one must calculate the residue for the product of four infinite series, namely, the expansions of $[\tau - A]^{-1/2}$, $[\tau - A]^{-1/2}$, $\ln\tau$ and $[\tau - z]^{-1}$ where $\tau = \mathrm{e}^{\mathrm{i}\xi}$, $z = \mathrm{e}^{\mathrm{i}\theta}$ and $A = \mathrm{e}^{\mathrm{i}c}$. For an odd function φ_-, which is the case in all problems considered in [1–3], the zero Fourier component Φ_{0-} should always equal zero, and this difficulty was not apparent at first, but for an even function $\varphi_-(\theta;\gamma)$ its resolve is unavoidable. For the time being, we make use of the expansion

$$\xi = \sum_{n=-\infty}^{\infty}{}' \frac{(-)^{n+1}}{\mathrm{i}n}\mathrm{e}^{\mathrm{i}n\xi}. \tag{3.10}$$

together with (3.8) and the fact that $\Phi_{n-}(\gamma)/n$ is odd to rewrite the solution (3.3) in the form

$$\begin{aligned}\varphi_-(\theta,\gamma) = -R(\theta)\Bigg[& Q_0(\gamma)\Phi_{0-}(\gamma)\sum_{n=1}^{\infty}\frac{(-)^n}{n}\{I_n(\theta) - I_{-n}(\theta)\} \\ & - \sum_{n=1}^{\infty} Q_n(\gamma)\frac{\Phi_{n-}(\gamma)}{n}\{I_n(\theta) - I_{-n}(\theta)\}\Bigg].\end{aligned} \tag{3.11}$$

Substituting in the formula

$$\Phi_{\ell-} = \frac{1}{2\pi}\int_{-c}^{c}\varphi_-(\theta)\mathrm{e}^{-\mathrm{i}\ell\theta}\,\mathrm{d}\theta \tag{3.12}$$

according to (3.10), we get the algebraic system

$$\Phi_{\ell-}(\gamma) + Q_0(\gamma)\Phi_{0-}(\gamma)N_\ell - \sum_{n=1}^{\infty} Q_n(\gamma)\frac{\Phi_{n-}(\gamma)}{n}[N_{n\ell} - N_{-n\ell}] = 0 \qquad \ell = 0, 1, 2, \ldots \tag{3.13}$$

where for $n \neq 0$ we have used the notation

$$\begin{aligned} N_{-n\ell} &= \frac{1}{2\pi}\int_{-c}^{c} R(\theta) I_{-n}(\theta)\mathrm{e}^{-\mathrm{i}\ell\theta}\,\mathrm{d}\theta \\ &= \sum_{j=0}^{n-1} A_{j-n-\ell-1}\sum_{m=0}^{j}\frac{(2m-1)!![2(j-m)-1]!!}{2m!!(2j-2m)!!\mathrm{e}^{\mathrm{i}(j-2m)c}} \\ N_{n\ell} &= -\sum_{j=0}^{n-1} A_{n-j-\ell-2}\sum_{m=0}^{j}\frac{(2m-1)!![2(j-m)-1]!!}{2m!!(2j-2m)!!\mathrm{e}^{-\mathrm{i}(j-2m)c}} \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} A_k &= \frac{1}{2\pi}\int_{-c}^{c} R(\theta)\mathrm{e}^{\mathrm{i}(k+1)\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\bar{A}A}\sqrt{(t-A)(t-\bar{A})}t^k\,\mathrm{d}t \qquad A = \mathrm{e}^{\mathrm{i}c} \end{aligned} \tag{3.15}$$

and the arc $\bar{A}A$ is directed in the positive sense along the unit circle in the complex t-plane. The values of these integrals are given by the formulae [3, p 216]

$$A_{-1} = \frac{e^{-ic} + e^{ic}}{4} - \frac{1}{2} = A_{-2},$$
$$A_{-k} = \frac{1}{2^k}\left[\frac{(2k-5)!![e^{-i(k-1)c} + e^{i(k-1)c}]}{(k-1)!} - \sum_{m=1}^{k-2} \frac{(2m-3)!!(2k-2m-5)!!}{m!(k-1-m)!e^{i(k-1-2m)c}}\right] \qquad k > 2 \tag{3.16}$$

and for $k \geqslant 0$ we have [1]

$$A_k = A_{-(k+3)}. \tag{3.17}$$

Finally, N_ℓ is given by

$$N_\ell = \sum_{n=1}^{\infty} \frac{(-)^n}{n}\{N_{n\ell} - N_{-n\ell}\} \qquad (\ell \in N^+). \tag{3.18}$$

It has been shown [1] that $N_{-\ell n}(N_{\ell n})$ tends to zero as $\ell \to \infty$ more rapidly than $1/\ell$ and indeed, it tends to zero as $n \to \infty$ since

$$\frac{1}{\ell}N_{\pm\ell n} = \frac{1}{n}N_{\pm n\ell}. \tag{3.19}$$

Thus, the coefficients of the Fourier components to the right-hand side of system (3.13) tend to zero as n and ℓ increase indefinitely. The coefficients N_ℓ of $\Phi_{0-}(\gamma)$ represent the technical difficulties still to be resolved. These infinite series need much effort to calculate and thereafter can still influence the accuracy of the solution. In order to sum up these series, we first note that the solution of system (3.13) must obey the condition

$$\varphi_-(\pi;\gamma) = \Phi_{0-}(\gamma) + 2\sum_{\ell=1}^{\infty} \cos \ell\pi\, \Phi_{\ell-}(\gamma) = 0 \tag{3.20}$$

since it is imposed as long as $c \in [0, \pi)$. On multiplying the ℓth equation, $\ell \in N$, of system (3.13) by $2\cos \ell\pi$ and adding to the first one, it is a simple matter to verify that the relation

$$[N_{n^0} - N_{-n^0}] + 2\sum_{\ell=1}^{\infty}(-)^\ell[N_{n\ell} - N_{-n\ell}] = 0 \qquad n \in N \tag{3.21}$$

is necessary and sufficient for condition (3.20) to be satisfied. Indeed, the last relations are fulfilled. For example, we have (recalling (3.17))

$$\begin{aligned}[N_{10} - N_{-10}] &- 2[N_{11} - N_{-11}] + 2[N_{12} - N_{-12}] - 2[N_{13} - N_{-13}] + \cdots \\ &= [-A_{-1} - A_{-2}] - 2[-A_{-2} - A_{-3}] + 2[-A_{-3} - A_{-4}] - 2[-A_{-4} - A_{-5}] \\ &\quad + \cdots = 0\end{aligned}$$

and so on. Finally, in view of (3.19) and (3.21) we have

$$\begin{aligned}N_\ell &= \sum_{n=1}^{\infty} \frac{(-)^n}{n}(N_{n\ell} - N_{-n\ell}) \\ &= \sum_{n=1}^{\infty} \frac{(-)^n}{\ell}(N_{\ell n} - N_{-\ell n}) \\ &= -\frac{1}{2\ell}[N_{\ell 0} - N_{-\ell 0}] \\ &= -\frac{1}{\ell}N_{\ell 0}\end{aligned} \tag{3.22}$$

where the last step follows from the relation $N_{n0} = -N_{-n0}$ which can be verified on letting $\ell = 0$ in both definitions (3.14), and bearing equation (3.17) in mind. In brief, system (3.13) can now be written in the form

$$\Phi_{0-} + N_0 Q_0(\gamma)\Phi_{0-}(\gamma) - \sum_{n=1}^{\infty} \frac{Q_n(\gamma)}{n}[N_{n0} - N_{-n0}]\Phi_{n-}(\gamma) = 0$$

$$\Phi_{\ell-}(\gamma) - \frac{1}{\ell} N_{\ell 0} Q_0(\gamma)\Phi_{0-}(\gamma) - \sum_{n=1}^{\infty} \frac{Q_n(\gamma)}{n}[N_{n\ell} - N_{-n\ell}]\Phi_{n-}(\gamma) = 0 \qquad \ell = 1, 2, 3. \tag{3.23}$$

This form is clearly more simplified than (3.13). The first equation of system (3.23), involving N_0, the sole coefficient not exactly calculated, represents no profound difficulty since it can be replaced by (3.20).

In the same way as followed in [1] it can be verified that the small eigenvalues may be obtained by truncating system (3.23) at suitable orders. Moreover, the calculations confirm this assertion as shown in section 6. The influence of the truncation on the eigenfunctions is discussed in section 5.

4. The uniform cases $c = 0, \pi$

It can be shown that the solution obtained by solving equation (3.23) at $c = 0, \pi$ coincides with the classical solutions of the Dirichlet and Neumann problem, respectively. Indeed, if $c = 0$, then all the integrals A_k involved by the definitions (3.13) of $N_{\pm n\ell}$ will vanish (see definition (3.14)). Consequently, system (3.23) will be reduced to

$$\Phi_{\ell-}(\gamma) = 0 \qquad \ell = 0, 1, 2, \ldots \tag{4.1}$$

and the solution (2.12) will be trivial except at those values of γ for which the denominator of any term on its right-hand side vanishes. For the sake of definiteness, in the event that γ_i is a zero of $J_{|\ell|}(\gamma)$ say, solution (2.12) will be reduced to the form

$$S(r, \theta; \gamma_i) \sim J_{|\ell|}(\gamma_i r)\mathrm{e}^{\mathrm{i}\ell\theta} \tag{4.2}$$

which is just a classical partial solution of the uniform Dirichlet problem. Thus, the eigenvalues of the problem in this case are the solution of

$$J_\ell(\gamma) = 0 \qquad \ell = 0, 1, 2, \ldots. \tag{4.3}$$

To verify the validity of the similar conclusion for the case $c = \pi$, we rewrite expression (3.14) in the form

$$N_{-n\ell} = \sum_{j=0}^{n-1} A_{j-n-\ell-1} S_j(c) \qquad N_{n\ell} = \sum_{j=0}^{n-1} A_{n-j-\ell-2} S_j(c). \tag{4.4}$$

It could be shown [1] that

$$S_j(0) = 1 \qquad j \in N^+. \tag{4.5}$$

Together with the definition of the inner summation $S_j(c)$, this leads to the result

$$S_j(\pi) = (-1)^j \tag{4.6}$$

but since in this case we have (definition (3.15))

$$A_k = \begin{cases} -1 & k = -1, -2 \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

it follows immediately that

$$N_{-n\ell} = 0 \qquad N_{n\ell} = \delta_{\ell n} \qquad N_{n0} = -N_{-n0} = (-)^n \qquad n, \ell \in N. \tag{4.8}$$

Thus, as c nears π indefinitely, system (3.23) becomes

$$\frac{(-)^\ell}{\ell} Q_0(\gamma)\Phi_{0-}(\gamma) + \left[1 - \frac{1}{\ell} Q_\ell(\gamma)\right] \Phi_{\ell-}(\gamma) = 0. \tag{4.9}$$

According to definition (2.16), this system may have a non-trivial solution only if γ is a zero of the relation

$$[\ell J_\ell(\gamma) - \gamma J_{\ell+1}(\gamma)] = 0 \qquad \ell = 0, 1, 2, \ldots \tag{4.10}$$

which is the well known result defining the eigenvalues of the uniform Neumann problem.

5. Justification of the truncation

As we have pointed out above, the constant α in the right-hand side of equation (3.2) is immaterial (see the inversion formula (3.3) together with (3.8) whatever be the value of α. See also the reduced system (3.23)) and can practically be set equal to zero. Thus, this equation is a homogeneous one, and the estimation of the error taking place in the solution of such an equation as a result of an approximation process cannot yet be found in the literature. However, the approximation used in this paper on obtaining a solution $\varphi_-(\theta; \gamma_i)$, namely replacing this function in equation (3.2) by the first few terms of its Fourier representation or equivalently the truncation of system (3.23), can be justified provided the designation of the corresponding γ_i can practically reach its infinite precision.

The solution $\varphi_-(\theta, \gamma_i)$ of the homogeneous equation (3.2) for which the zero Fourier component $\Phi_{0-}(\gamma_i)$ equals 1 is clearly the solution of the inhomogeneous equation

$$K\varphi = f \tag{5.1}$$

where

$$K\varphi_- = \frac{1}{\pi \mathrm{i}} \int_{-c}^{c} \frac{\varphi_-(t, \gamma_i)}{1 - \mathrm{e}^{\mathrm{i}(\theta - t)}}\, \mathrm{d}t - \sum_{n=-\infty}^{\infty}{}' Q_{|n|}(\gamma_i)\Phi_{n-}(\gamma_i) \frac{\mathrm{e}^{\mathrm{i}n\theta}}{\mathrm{i}n} \tag{5.2}$$

and

$$f = -Q_0(\gamma_i) \sum_{n=-\infty}^{\infty} \frac{(-)^n}{\mathrm{i}n} \mathrm{e}^{\mathrm{i}n\theta}. \tag{5.3}$$

Indeed, following the same steps as in section 3, the above equation can be reduced to the inhomogeneous algebraic system

$$\begin{aligned} &\Phi_{0-}(\gamma_i) - \sum_{n=1}^{\infty} \frac{Q_n(\gamma_i)}{n} [N_{n0} - N_{-n0}]\Phi_{n-}(\gamma_i) = -Q_0(\gamma_i) N_0 \\ &\Phi_{\ell-}(\gamma_i) - \sum_{n=1}^{\infty} \frac{Q_n(\gamma_i)}{n} [N_{n\ell} - N_{-n\ell}]\Phi_{n-}(\gamma_i) = \frac{1}{\ell} N_{\ell 0} Q_0(\gamma_i) \qquad \ell \in N. \end{aligned} \tag{5.4}$$

Further, the determinant of this system does not vanish since its truncation at any order differs from the corresponding one of system (3.23) which finally vanishes at γ_i. Clearly systems (3.23) and (5.4), both truncated at arbitrary order $\ell = j$ (say), will have the same solution. To solve system (3.23), we first set $\Phi_0(\gamma_i) = 1$ in the last j-equations and then, after solving them, the solution will satisfy the first equation automatically, while the last j-equations of system (5.4) are identical to those mentioned above and therefore will have the same solution for $\Phi_\ell(\gamma_i)$, $\ell = 1, 2, \ldots, j$. The substitution of this solution in the first equation of system (5.4), comparing it with the first equation of system (3.23), will yield the inevitable result $\Phi_{0-}(\gamma_i) = 1$.

Having come to the conclusion that equation (3.2), as γ assumes an eigenvalue γ_i, is equivalent to the inhomogeneous equation (5.1), the truncation can now be justified as follows. On truncating system (3.23) at the jth order, we approximate equation (5.1) to the more simple form

$$\tilde{K}\tilde{\varphi}_- = f \tag{5.5}$$

where $\tilde{K}$ and $\tilde{\varphi}$ are defined by replacing the infinity symbol in equations (5.2) and (5.3) by the bounded value j. Taking into consideration that the Banach space $L_2[-c, c]$ is the domain and the range of both the operators K and $\tilde{K}$, the norm of the operator $K - \tilde{K}$ can easily be estimated. Indeed we have

$$\|(K - \tilde{K})\varphi_-(x)\| = \int_{-c}^{c} \left| \frac{1}{\pi} \sum_{n=j}^{\infty} \frac{Q_n}{n} \int_{-c}^{c} \sin n(x - t)\varphi_-(t)\,\mathrm{d}t \right|^2 \mathrm{d}x \leqslant 4\left(\sum_{n=j}^{\infty} \frac{Q_n}{n} \right)^2 \|\varphi_-\|^2 \tag{5.6}$$

and consequently,

$$\|K - \tilde{K}\| \leqslant 2\sum_{n=j}^{\infty} \frac{Q_n}{n}. \tag{5.7}$$

The right-hand side in equation (5.7) tends to zero since $Q_n = \mathrm{O}(1/n)$ according to definition (2.16). Together with the fact that equation (5.5) has a unique bounded solution (3.10) truncated at the jth order and in which $\{1, \Phi_{n-}(\gamma_i); n = 1, 2, \ldots, j\}$ is the solution of system (5.4) truncated at the jth order, equation (5.7) leads to the boundedness of the operator $\tilde{K}^{-1}(K - \tilde{K})$ and its norm eventually satisfies

$$\|\tilde{K}^{-1}(K - \tilde{K})\| < 1 \tag{5.8}$$

provided j is sufficiently large. In view of this result [5], it follows that equation (5.1), or equivalently (3.2) in which $\gamma = \gamma_i$, has the unique solution

$$\varphi_-(\gamma_i) = \tilde{\varphi}_-(\gamma_i) + [I + \tilde{K}^{-1}(K - \tilde{K})]^{-1}\tilde{K}^{-1}(f - K\tilde{\varphi}(\gamma_i)) \tag{5.9}$$

where I is the unit operator. Further, the resulting error due to the truncation can be estimated according to the formula

$$\|\varphi_- - \tilde{\varphi}_-\| \leqslant \frac{\|\tilde{K}^{-1}(f - K\tilde{\varphi})\|}{1 - \|\tilde{K}^{-1}(K - \tilde{K})\|}. \tag{5.10}$$

Indeed, the calculations have been continued right to the designation of the eigenvalues γ_i, $i = 1, \ldots, 6$, as well as the corresponding sets $\{\Phi_{n-}(\gamma_i), n \in N^+$ and $|n| \leqslant 70\}$ at several values of the parameter c. In all these cases, the numerical results strongly suggest the validity of inequality (5.8) as soon as j grows beyond relatively small limits. The situation is illustrated in section 6.

6. Numerical verification

Some numerical results are shown here to give an idea about the influence of the truncation on the eigenvalues and their corresponding eigenfunctions of the problem and therefore represent an investigation of the usefulness of the method. On acquiring these results, system (3.23) was truncated at different orders: $j = 4, 5, \ldots$. The eigenvalues at different values of the parameter c were then obtained through equating the determinants of the truncated system to zero. At a certain value of c, $\gamma_i^{(j)}$ will stand for the ith zero of the determinant truncated at the jth order. The greater the increase in j is the greater the number of decimal places in which the eigenvalues remain unchanged. The order j at which the eigenvalue becomes practically precise depends on the value of the parameter c as well as the order i of the eigenvalue itself. To illustrate, table 1 exhibits the first six eigenvalues in the case $c = \frac{1}{2}\pi$. The first eigenvalue is stable to four decimal places from the eighth-order onwards while the second and the third reach the same degree of precision at the 10th and 12th order, respectively. For the next three eigenvalues, j should be increased considerably. In fact, the right-hand side of (3.23), viewed as an expansion of $\Phi_{\ell-}(\gamma)$ decays slowly as γ increases since $Q_n(\gamma) = 0(\gamma^2)$ (definition (2.16)) so that more terms will be required for the validity of the approximation of $\Phi_{\ell-}(\gamma)$, $\ell \in N^+$, as well as the accuracy of the representation of $\varphi_-(\theta; \gamma)$ and the precision of the corresponding eigenvalue γ thereby. In addition to the case of a large eigenvalue, higher orders of the truncation are also required when c becomes small. Indeed, the calculations show that as c decreases (increases) below (above) $\frac{\pi}{2}$, the higher (lower) the order j we need to realize a precision similar to that in table 1. This can simply be traced back to the fact that as the support of a function φ_- becomes narrower, a large number of its Fourier representation are needed to approximate it conveniently. At this stage it is appropriate to exhibit in table 2 the evolution of the first and second eigenvalues as c varies from 0 to π. The monotonic variation is to be noted.

Table 1. The first six eigenvalues in the case $c = \frac{1}{2}\pi$.

i	1	2	3	i	4	5	6
$\gamma_i^{(8)}$	1.2444	2.9433	4.6049	$\gamma_i^{(17)}$	5.4413	6.1457	7.3053
$\gamma_i^{(9)}$	1.2444	2.9433	4.6049	$\gamma_i^{(18)}$	5.4413	6.1457	7.3053
$\gamma_i^{(10)}$	1.2444	2.9432	4.6049	$\gamma_i^{(19)}$	5.4412	6.1457	7.3053
$\gamma_i^{(20)}$	1.2444	2.9432	4.6048	$\gamma_i^{(20)}$	5.4412	6.1457	7.3053

Table 2. The first two eigenvalues at $c = \frac{s}{6}\pi$, $s = 0, 1, \ldots, 6$.

	c						
i	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
1	2.4048	2.1623	1.6108	1.2444	0.9914	0.7767	0.0000
2	3.8317	3.1953	2.9580	2.9432	2.7480	2.4084	1.8412

Table 3. The components $\Phi_{\ell-}(\gamma_1)$, $\ell = 1, 2, \ldots, 20$; in the case $c = \frac{\pi}{2}$ taken from top left, respectively.

0.7325	0.2072	−0.0983	−0.0664	0.0448	0.0353	−0.0269	−0.0227	0.0184	0.0161
−0.0136	−0.0122	0.0106	0.0097	−0.0086	−0.0079	0.0071	0.0066	−0.0060	−0.0056

The Fourier components $\Phi_{\ell-}(\gamma_i)^{(j)}$, $\ell = 1, 2, \ldots, j$ can be obtained on substituting $\gamma = \gamma_i^{(j)}$ in system (3.23) truncated at the jth order, setting $\Phi_{0-}(\gamma_i) = 1$, and then solving

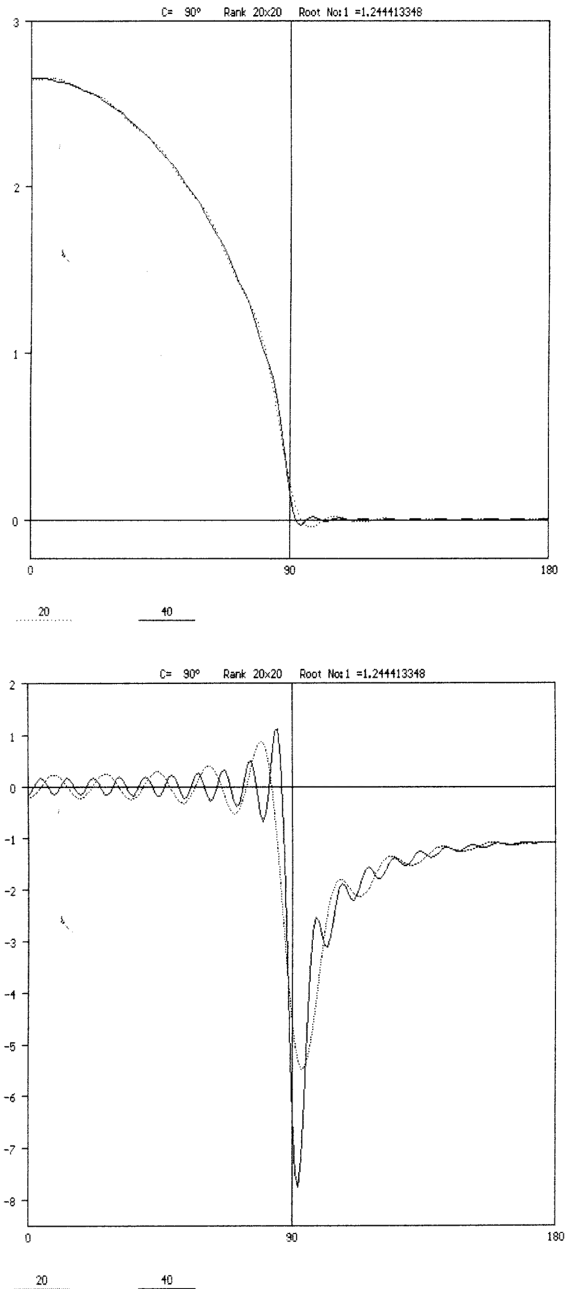

Figure 1. (Top) The function $\varphi_{-}(\theta;\gamma_1)$ in the case $c=\frac{1}{2}\pi$ truncated at orders 20 (dotted curve) and 40 (full curve). (Bottom) The function $\varphi_{+}(\theta;\gamma_1)$ in the case $c=\frac{1}{2}\pi$ truncated at orders 20 (dotted curve) and 40 (full curve).

any j of the resulting equation, provided their determinant does not vanish. The remaining equation will be automatically satisfied. Further, the approximation

$$\Phi_{\ell-}(\gamma_i^{(j)}) = \frac{1}{\ell} N_{\ell 0} Q_0(\gamma_i^{(j)}) + \sum_{n=1}^{j} \frac{Q_n(\gamma_i^{(j)})}{n} [N_{n\ell} - N_{-n\ell}] \Phi_{n-}(\gamma_i^{(j)})$$
$$\ell = j+1, j+2, \ldots, k. \tag{6.1}$$

can be used to obtain more components of the function $\varphi_-^{(j)}(\theta; \gamma_i)$. At sufficiently large j, this approximation provides the same results which would be obtained through increasing the order of the truncation j. For example, recalling that in the case $c = \frac{\pi}{2}$ the first eigenvalue $\gamma_1^{(j)}$ (see table 1) remains unchanged from $j = 8$ onwards, thus the first 20 components of the corresponding solution can be obtained through 13 different truncations, namely those for which $j = 8, 9, \ldots, 20 = k$. For all these processes, the calculations provide the components unchanged to four decimal places (see table 3). In fact, according to the approximation (6.1), the first neglected term in the expression of $\Phi_{(j+1)-}(\gamma^{(j)})$ is simply

$$\frac{Q_n(\gamma_i^{(j)})}{n} [N_{n\ell} - N_{-n\ell}] \Phi_{n-}(\gamma_i^{(j)}) \Big|_{n=\ell=j+1} < 0(j+1)^{-4} \tag{6.2}$$

which confirms the high validity of approximation (6.1). In this estimation, use has been made of the relations (2.16), (3.19) as well as the asymptotic behaviour of the quantities $N_{\pm n\ell}$ mentioned in the discussion preceding (3.19), and the fact that Φ_{n-} is a Fourier component of a continuous function. Thus, for the left-hand side of (5.10), we have $\|\varphi_- - \tilde{\varphi}_-(j)\| \leqslant 0.0000\ldots$, $j \geqslant 8$ where $\tilde{\varphi}_-(j)$ stands here for the function(s) $\tilde{\varphi}_-(\theta; \gamma_1^{(j)})$ while decreasing j to 7, the resulting error becomes $\|\varphi_- - \tilde{\varphi}_-(7)\| \approx 0.0006\ldots$.

Finally, to demonstrate the usefulness of the method, figures 1(a) and 1(b) show the plots of the functions $\varphi_\pm(\theta, \gamma_1)$ defined according to the formulae

$$\varphi_-(\theta, \gamma_1) \approx 1 + 2 \sum_{n=1}^{k} \Phi_{n-}(\gamma_1) \cos n\theta \qquad k = 20, 40 \tag{6.3}$$

$$\varphi_+(\theta; \gamma_1) = \frac{\partial S(1, \theta; \gamma_1)}{\partial r} \approx -Q_0(\gamma_1) + 2 \sum_{n=1}^{k} [n - Q_n(\gamma_1)] \Phi_n(\gamma_1) \cos n\theta \qquad k = 20, 40. \tag{6.4}$$

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References

[1] Eckhardt U and El Sheikh M G 1987 A Fourier method for initial value problems with mixed boundary conditions *Comput. Math. Applic.* **14** 189–99

[2] Chersky J I 1961 The reduction of periodic problems of mathematical physics to singular integral equations with Cauchy's kernel *Dokl. Akad. Nauk. SSSR* **140** 69–72

[3] Gakhov F D and Chersky J I 1978 *The Equations of Convolution Type* (Moscow: Nauka) p 215 (in Russian)

[4] Muskhelishvili N I 1953 *Singular Integral Equations* (Groningen: Noordhoff)

[5] Chersky J I 1963 Two theorems on estimation of the error and some of their applications *Dokl. Akad. Nauk. SSSR* **150** 271–4